



Numerical modelling of qualitative behaviour of solutions to convolution integral equations

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Abstract

We consider the qualitative behaviour of solutions to linear integral equations of the form

$$y(t) = g(t) + \int_0^t k(t-s)y(s) \, ds, \quad (1)$$

where the kernel k is assumed to be either integrable or of exponential type. After a brief review of the well-known Paley–Wiener theory we give conditions that guarantee that exact and approximate solutions of (1) are of a specific exponential type. As an example, we provide an analysis of the qualitative behaviour of both exact and approximate solutions of a singular Volterra equation with infinitely many solutions. We show that the approximations of neighbouring solutions exhibit the correct qualitative behaviour.

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1. Introduction

We are concerned with preserving qualitative properties of exact solutions of equations under discrete (numerical) approximations. These qualitative features include (but are not restricted to) asymptotic behaviour of the solution set to a problem for large time. Much of the existing literature is concerned with solutions that converge to zero as $t \rightarrow \infty$ or which are bounded as $t \rightarrow \infty$. However, it can be equally important to consider the effectiveness of numerical schemes in reproducing other types of behaviour. Thus, in recent work we have been concerned [7,6,9–12] with periodic or oscillatory behaviour of solutions, and with the preservation of bifurcation points in numerical solutions of delay and integral equations.

This paper is concerned with equations where we are able to deduce results about *rates of growth or decay*. We develop analytical results for the exact solutions of some convolution integral equations and for their numerical approximations and we show how our results provide insights into the exact and numerical solution of some singular integral equations of interest from the literature.

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For linear equations, such as (1) there can be a close correspondence between qualitative behaviour of solutions and stability of a particular solution under small perturbations of the problem. For example, consider the perturbation of Eq. (1) caused by changing the forcing function g to \tilde{g} . (We assume the restriction that $g, \tilde{g} \in \mathcal{G}$, some linear space of permitted (*admissible*) forcing functions.) The difference in the two solutions now satisfies an equation of form (1) with the forcing function g replaced by $\tilde{g} - g \in \mathcal{G}$. Now if all solutions to (1) for $g \in \mathcal{G}$ are bounded or tend to zero, then so will be the perturbation caused by the change in g . Thus, if all solutions to (1) are bounded, then all will be stable with respect to changes in the forcing term; if all tend to zero for large t then all will be asymptotically stable with respect to perturbations of this type.

2. Preliminaries

As is customary, we shall use the notation $L^p(\mathbb{R}^+)$ to represent the class of Lebesgue measurable functions for which $\int_0^\infty |f(t)|^p dt < \infty$. In practice, it may be more appropriate to restrict our functions to have a finite number of discontinuities. The class $BC(\mathbb{R}^+)$ will be used for the bounded continuous functions on the positive real line and the class $PC(\mathbb{R}^+)$ will refer to the set of piecewise continuous functions.

For our discussion of qualitative behaviour of solutions (see the next section) it will be convenient to introduce the idea of *functions of exponential type*.

Definition 2.1. A function f is of exponential order as $t \rightarrow \infty$ if there exist constants A, α such that $|f(t)| \leq Ae^{\alpha t}$ for all $t \geq 0$. A measurable function f is of exponential type if there exists some constant α for which $f(t)e^{-\alpha t} \in L^1(\mathbb{R}^+)$. A measurable function f is of exponential type σ if there exists a constant σ for which $f(t)e^{-\sigma t} \in L^1(\mathbb{R}^+)$.

Remark 2.1. Note that if we define $s = \inf\{\sigma: f \text{ is of exponential type } \sigma\}$ then f need not necessarily be of exponential type s .

Definition 2.2. For two functions x, y defined on \mathbb{R}^+ the convolution product is defined by

$$x * y(t) = \int_0^t x(t-s)y(s) ds \quad (2)$$

for every value of $t \geq 0$ for which the integral exists.

This provides us with convenient notation that enables us to write Eq. (1) in the form

$$y(t) = g(t) + k * y(t). \quad (3)$$

For measurable functions f of exponential type σ , the Laplace transform, denoted

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt \quad (4)$$

exists and is finite for $\Re(s) \geq \sigma$.

Properties of the convolution that will be of interest in this paper are contained in the following theorem and corollary. Further details can be found in [4,13].

Theorem 2.1. (1) $\mathcal{L}[x * y](s) = \mathcal{L}[x](s)\mathcal{L}[y](s)$ providing all the Laplace transforms exist.

(2) If $x \in L^1(\mathbb{R}^+)$ and $y \in L^p(\mathbb{R}^+)$ then $x * y \in L^p(\mathbb{R}^+)$.

Corollary 2.1. (1) If x, y are measurable functions of exponential type σ then $x * y$ is of exponential type σ .

(2) If x, y are bounded continuous functions then $x * y$ is of exponential order.

Remark 2.2. Part (i) of the corollary can be proved by direct substitution into the definition of convolution and part (ii) follows from part (i) by considering all values of $\sigma > 0$.

For some of our work, it turns out that the property of positive definiteness will be important. We will use the following important characterisation of positive definite functions, due to Bochner.

Definition 2.3. A function f on $[0, \infty)$ is positive definite if and only if $\mathcal{L}[f](s) \geq 0$ for all $s : \Re(s) > 0$.

We shall combine the notions of positive definiteness and exponential type in the following way:

Definition 2.4. A function F will be said to be *positive definite of exponential type* $\sigma > 0$ if there exists a positive definite function $f \in L^1[0, \infty)$ for which $F(t)e^{-\sigma t} = f(t)$.

3. The resolvent kernel

For Eq. (3) one can propose an iterative approach to its solution. For the moment, we proceed formally, and let

$$y_0(t) = g(t),$$

$$y_n(t) = g(t) + k * y_{n-1}(t), \quad n = 1, 2, 3, \dots$$

Correspondingly, we define

$$r_1(t) = k(t),$$

$$r_n(t) = k(t) + k * r_{n-1}(t), \quad n = 2, 3, \dots$$

It follows that

$$y_n(t) = g(t) + r_n * g(t). \quad (5)$$

So, assuming all the convolutions exist, and that $\{r_n\}$ is a convergent sequence of functions on \mathbb{R}^+ with limit function r , then it follows that $\{y_n\}$ converges to a function y satisfying

$$y(t) = g(t) + r * g(t), \quad (6)$$

where r satisfies

$$r = k + k * r. \quad (7)$$

Any function r that satisfies (7) (however, it might be derived) is known as a resolvent kernel for (3). The solution of (3) is then given by formula (6).

If we assume $g, k \in L^1(\mathbb{R}^+)$ and that the solution y has a Laplace transform, and take Laplace transforms in (3) we obtain

$$\mathcal{L}[y](s) = \mathcal{L}[g](s) + \mathcal{L}[k](s)\mathcal{L}[y](s). \quad (8)$$

If $\mathcal{L}[k](s) \neq 1$, a simple rearrangement of (8) gives

$$\mathcal{L}[y](s) = \frac{\mathcal{L}[g](s)}{1 - \mathcal{L}[k](s)} = \mathcal{L}[g](s) \left(1 + \frac{\mathcal{L}[k](s)}{1 - \mathcal{L}[k](s)} \right), \quad (9)$$

which leads to the conclusion that a resolvent kernel satisfies

$$\mathcal{L}[r](s) = \frac{\mathcal{L}[k](s)}{1 - \mathcal{L}[k](s)}. \quad (10)$$

The classical Paley–Wiener theorem (see [20]) provides a firm theoretical basis for this analysis by guaranteeing the existence of a suitable resolvent kernel r .

Theorem 3.1 (Paley–Wiener). If $L(s)$ is the Laplace transform of a function in $L^1(\mathbb{R}^+)$ and $\varphi(u)$ is analytic over the range of values of L and includes 0 in its range then $\varphi(L(s))$ is the Laplace transform of some function in $L^1(\mathbb{R}^+)$.

By setting $\varphi(u) = u/(1 - u)$ one can deduce (see [3]):

Corollary 3.1. Let $k \in L^1(\mathbb{R}^+)$ satisfy $\mathcal{L}[k](s) \neq 1$ for $\Re(s) \geq 0$ then the resolvent kernel r exists and $r \in L^1(\mathbb{R}^+)$.

Now, for kernels of exponential type it is possible to undertake essentially the same analysis (see below). One needs merely to take care to ensure that the Laplace transforms used in the analysis exist in a suitable half-plane. We recall that if the function f is of exponential type α then the Laplace transform $\mathcal{L}[f](s)$ exists and is analytic in the half plane $\Re(s) > \alpha$. As we saw in Corollary 3.1, if the denominator of the expression for $\mathcal{L}[y](s)$ does not vanish for $\Re(s) \geq 0$ then the resolvent kernel exists and is integrable. We turn our attention now to the case where the denominator $1 - \mathcal{L}[k](s)$ vanishes for some s : $\Re(s) \geq 0$.

We give the following theorem (see, for example, [13, p. 46, Corollary 4.2]):

Theorem 3.2. Assume that

$$(H1) \quad k \in L^1(\mathbb{R}^+) \quad \text{and} \quad 1 - \mathcal{L}[k](s) = 0 \quad \text{if and only if} \quad s \in \mathcal{S} \subset \mathbb{C},$$

where $\sigma > 0$ is chosen so that $\sigma > \Re(s)$ for every $s \in \mathcal{S}$, then $(\mathcal{L}[k](s)/(1 - \mathcal{L}[k](s)))$ is the Laplace transform of some function of the form $e^{\sigma t} \tilde{f}(t)$ where $\tilde{f} \in L^1(\mathbb{R}^+)$.

Remark 3.1. Hypothesis (H1) requires that the kernel k be integrable. However, if instead we assume that the kernel k is of exponential type $\alpha > 0$, then the conclusions of Theorem 3.2 apply so long as σ in (H1) is chosen so that additionally $\sigma > \alpha$.

The proof of Theorem 3.2 is straightforward and follows from the Paley–Wiener theorem above and the translation property for Laplace transforms. We observe that $h(s) = 1/(1 - \mathcal{L}[k](s))$ is analytic to the right of $\Re(s) = \sigma$ and we set $\tilde{h}(s) = h(s + \sigma)$. The resulting function \tilde{h} is analytic on the right half plane and so by the Paley–Wiener theorem it follows that \tilde{h} is the Laplace transform of some function $\tilde{f} \in L^1(\mathbb{R}^+)$. The translation property of Laplace transforms (see [4] for example) now leads (since $h(s) = \tilde{h}(s - \sigma)$) to the conclusion of the theorem.

From the above analysis, it follows that we may be interested in the solution of Volterra equations where the resolvent kernel is of exponential type σ rather than integrable. We consider the behaviour of the solution of the equation in this case.

Proposition 3.1. Assume that $g \in L^1(\mathbb{R}^+)$ and r is of exponential type $\sigma \geq 0$. The convolution $g * r$ is then of exponential type σ .

Proof. We know that the Laplace transforms of g and r are analytic functions, respectively, on the right half plane and on $\Re(s) > \sigma$. It follows that the product of the transforms of g, r is analytic on $\Re(s) > \sigma$. Hence the product of the transforms of g and r is the Laplace transform of a function y of exponential type σ . Finally, we observe that since all these transforms exist, we can use the uniqueness theorem for Laplace transforms to deduce that $g * r = f$ almost everywhere. \square

The above proposition means that we can now give some idea of the long-term behaviour of solutions to (3) in the case we have considered here. If g is integrable and the kernel k satisfies (H1) then the solution y is of exponential type σ . It follows that $e^{-\sigma t} y(t) \rightarrow 0$ almost everywhere as $t \rightarrow \infty$.

For kernels that are positive definite of exponential type $\sigma > 0$ we can go a little further using Bochner's characterisation of positive definiteness. We assume:

$$(H2a) \quad k(t) = \lambda \mathcal{K}(t) e^{\sigma t},$$

$$(H2b) \quad \mathcal{K} \in L^1[0, \infty) \text{ is positive definite and } \mathcal{K}(0) = 1, \text{ and}$$

$$(H2c) \quad g(t) = \mathcal{G}(t) e^{\sigma t} \text{ where } \mathcal{G} \in L^1[0, \infty),$$

k then satisfies $\mathcal{L}[k](s) \neq 1$ for every $\lambda : \Re(\lambda) < 0$ whenever $\Re(s) \geq \sigma$. It follows by the argument of this section that the solution y of (1) exists for every $\lambda : \Re(\lambda) < 0$ and satisfies $y(t) = \mathcal{Y}(t) e^{-\sigma t}$ where $\mathcal{Y} \in L^1[0, \infty)$.

Remark 3.2. We note further that \mathcal{Y} satisfies (1) with $g = \mathcal{G}$ and $k = \lambda \mathcal{K}$.

4. Application to a singular integral equation

In this section we analyse the equation (with singular kernel)

$$y(t) = g(t) + \int_0^t \frac{s^{\mu-1}}{t^\mu} y(s) \, ds, \quad (11)$$

where the forcing function $g \in PC(\mathbb{R}^+)$. This equation has been the subject of several papers (see, for example, [14,16–18,21]) and is particularly interesting because of the non-uniqueness of the solution in the case $0 < \mu < 1$. This non-uniqueness arises because, close to the origin, the kernel becomes unbounded. It can be shown (see, for example [18]) that the integral equation has a family of solutions, of which one is smooth, and all others have infinite gradient at $t = 0$. However, for $t > \varepsilon > 0$ all the solutions are smooth and therefore one would like to find a way to analyse the long-term behaviour of any particular solution.

For the analysis of the previous section to apply we need to express Eq. (11) in the form of a convolution equation. We put $\varsigma = \log(s)$ and $\tau = \log(t)$ so that we consider the equation:

$$y(e^\tau) = g(e^\tau) + \int_{-\infty}^{\tau} e^{\mu(\varsigma-\tau)} y(e^\varsigma) \, d\varsigma. \quad (12)$$

Now we put, for $\tau > 0$, $\gamma(\tau) = g(e^\tau) + e^{-\mu\tau} \int_{-\infty}^0 e^{\mu\varsigma} u(\varsigma) \, d\varsigma$ and $u(x) = y(e^x)$ so that (12) takes the form:

$$u(\tau) = \gamma(\tau) + \int_0^{\tau} e^{-\mu(\tau-\varsigma)} u(\varsigma) \, d\varsigma. \quad (13)$$

The function γ includes all the information about $y(t)$ for $t \in (0, 1]$ and this can be used (see [5]) to specify a particular one of the infinitely many solutions of (11). The Laplace transform-based analysis works on the solutions u (all smooth) of (13) for $\tau > 0$ (or, with respect to the original variables, for $t > 1$). In other words we have constructed a convolution equation (13) that has a unique smooth solution. Each of the solutions (smooth or non-smooth) of Eq. (11) gives rise to its own version of Eq. (13) through the particular function γ . However, the key information that enables us to analyse the asymptotic behaviour of y as $t \rightarrow \infty$ is the fact that the kernel of (13) is $e^{-\mu(\tau-\varsigma)}$.

First, we consider the properties of the function γ given by the expression:

$$\gamma(\tau) = g(e^\tau) + e^{-\mu\tau} \int_{-\infty}^0 e^{\mu\varsigma} u(\varsigma) \, d\varsigma. \quad (14)$$

The integral in the second term takes into account the (assumed given) initial values of y over $(0, 1]$ (correspondingly of u over $(-\infty, 0]$) and if we assume the integral to be finite then γ will be integrable whenever g is integrable.

Now we turn our attention to the convolution kernel of (13), whose Laplace transform is $1/(\mu + s)$. It follows from the analysis of the previous sections that:

- (1) If $\Re(\mu) > 1$ then $1/(\mu + s) \neq 1$ for $\Re(s) \geq 0$ and so the resolvent kernel $R \in L^1(\mathbb{R}^+)$.
- (2) If $0 < \Re(\mu) \leq 1$ then $1/(\mu + s) \neq 1$ for $\Re(s) > 1 - \Re(\mu)$ and so the resolvent kernel R is of exponential type $1 - \Re(\mu)$.

If we combine these observations about γ and R we can deduce the following results about the solutions to (13):

- (1) If $\Re(\mu) > 1$ then every solution of (13) is asymptotically stable (as $t \rightarrow \infty$) with respect to small changes in γ .
- (2) If $0 < \Re(\mu) \leq 1$ then small changes in γ lead to solutions that differ by at most an exponential growth term of order $1 - \Re(\mu)$ as $t \rightarrow \infty$.

As we pointed out earlier, the function γ contains both the forcing function g of the original equation (11) and details about the initial trajectory of the chosen solution. Therefore, these results provide us with information about the ways in which neighbouring solution trajectories of Eq. (11) will propagate as $t \rightarrow \infty$ as well as the effect of changing the forcing function g .

We will discuss the impact of this on the numerical results in a later section. For the present, we note that the paper [14] gives an explicit formula (used also in [18]) for the solution to (11) in the form:

$$y(t) = ct^{1-\mu} + g(t) + \frac{g(0)}{\mu-1} + t^{1-\mu} \int_0^t s^{\mu-2} (g(s) - g(0)) ds; \quad 0 < \mu < 1, \quad (15)$$

where c is an arbitrary constant that determines which of the solutions is under consideration. It is clear from this expression that the difference between neighbouring solutions y (that is, the difference between solutions with different values of c) propagates, as $t \rightarrow \infty$, at exponential rate $1 - \Re(\mu)$ which is in agreement with our theoretical result. Further, the exact solution in the case $\mu > 1$ does not have the $t^{1-\mu}$ growth term. Again this is in accordance with our results.

5. Numerical methods

In order to consider the behaviour of numerical methods applied to Volterra integral equations it is useful to derive discrete variants of the continuous theory. Essentially, one is concerned to solve an equation of the form

$$y_n = f_n + \sum_{j=0}^n a_{n-j} y_j, \quad n \geq 0 \quad (16)$$

and the corresponding approach (based on resolvents) would be to find a sequence $\{r_n\}$ that satisfies the resolvent equation

$$r_n = a_n + \sum_{s=0}^n a_{n-s} r_s \quad (17)$$

and to use the resolvent to express the solution to the discrete equation in the form:

$$y_n = f_n + \sum_{s=0}^n r_{n-s} f_s. \quad (18)$$

For details of this approach, see for example the works by Henrici [15], Nevanlinna [2] and Lubich (see below) and the references given in Baker [1]. The natural tools of analysis for discrete equations are the Z-transform, the formal power series or symbol, or the generating function. Any of these can be considered as a discrete tool that provides analogous power to the Laplace transform from the continuous case. More details are contained in, for example [8].

For the classical Paley–Wiener theory, the discrete analogue is discussed in the work of Lubich [19]. In this section we adapt the original theorem to give us the corresponding results for sequences of exponential type.

We begin by quoting the original result (see, for example [19]):

Theorem 5.1. *Consider the Volterra discrete equation (16) where the kernel $\{a_n\} \in \ell^1$. Then*

$$y_n \rightarrow 0 \quad \text{whenever } f_n \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad (19)$$

if and only if

$$\sum_{n=0}^{\infty} a_n \zeta^n \neq 1 \quad \text{for } |\zeta| \leq 1. \quad (20)$$

We extend this theorem to give the following result:

Theorem 5.2. *Consider the Volterra discrete equation*

$$\psi_n = \phi_n + \sum_{j=0}^n b_{n-j} \psi_j, \quad n \geq 0, \quad (21)$$

where the kernel $\{b_n\} \in \ell^1$. Then

$$\psi_n \beta^n \rightarrow 0 \quad \text{whenever } \phi_n \beta^n \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad (22)$$

if and only if

$$\sum_{n=0}^{\infty} b_n \zeta^n \neq 1 \quad \text{for } |\zeta| \leq \beta. \quad (23)$$

Proof. We observe that

$$\sum_{n=0}^{\infty} b_n \zeta^n \neq 1 \quad \text{for } |\zeta| \leq \beta \quad (24)$$

implies that

$$\sum_{n=0}^{\infty} b_n \beta^n \zeta^n \neq 1 \quad \text{for } |\zeta| \leq 1 \quad (25)$$

and therefore by setting $a_n = \beta^n b_n$, $y_n = \beta^n \psi_n$ and $f_n = \beta^n \phi_n$ we can apply Theorem 5.1 to yield the conclusion of the theorem. \square

Remark 5.1. (1) Note that, for $\beta > 1$ Theorem 5.2 provides stronger information than before about the speed of convergence of the sequence $\{y_n\}$ to zero as $n \rightarrow \infty$. On the other hand, for values of $\beta < 1$ it provides the result for discrete equations analogous to Theorem 3.2.

(2) In the continuous case we considered the effect of non-integrability of the kernel k . For the discrete equation we can extend the analysis in the same way: Let the sequence $\{k_n\}$ satisfy $\sum_{n=0}^{\infty} \alpha^n |k_n| < \infty$ for some $\alpha > 0$ and choose β in Theorem 5.2 so that $\beta < \alpha$. Then the conclusions of the theorem hold.

In [19] the author goes on to show that applying an A -stable linear multistep method (see, for example [3,22] for more details) to solve a Volterra equation with positive definite convolution kernel leads to a discrete scheme that preserves the stability of the original problem. Here we are able to provide a similar result, but we have to be careful.

To give the general idea we assume that we are solving (3), with a piecewise continuous kernel satisfying (H1), by a numerical scheme (for example a reducible quadrature based on a consistent and zero-stable—and hence convergent—linear multistep formula with convolution quadrature weights $\{\omega_n\}$) leading to a discrete scheme of the form:

$$y_n = f_n + h \sum_{j=0}^n \omega_{n-j} k_{n-j} y_j. \quad (26)$$

By (H1) we know that

$$\int_0^{\infty} e^{-ts} k(t) dt \neq 1 \quad (27)$$

for $\Re(s) > \sigma$. The method needs the use of special non-convolution starting weights which will be bounded and will contribute to the sequence $\{f_n\}$. We choose some fixed value of s for which $\Re(s) > \sigma$ so integral (27) is not equal to unity. Consider the expression (cf. Theorem 5.2)

$$\mathcal{L}_h(s) := \sum_{n=0}^{\infty} \kappa_n \zeta^n \quad \text{where } \zeta = \exp(-sh) \quad \text{and } h\omega_n k_n = \kappa_n. \quad (28)$$

Since k is of exponential type σ , $\sum_{n=0}^{\infty} |k_n| e^{-\sigma hn} < \infty$, further the sequence $\{\omega_n\}$ is bounded, so, for $\Re(s) > \sigma$, series (28) converges.

There clearly exists \hat{h} such that $h_0\omega_0k_0 < 1$ whenever $0 < h_0 < \hat{h}$ and so $\kappa_0 < 1$. Expression (28) can be viewed as an approximation (based on the convolution quadrature rule with weights $\{\omega_n\}$) for the integral in (27) and so one would expect, for a family of quadratures that converges to the true integral as $h \rightarrow 0$, that for small enough $h > 0$ expression (28) will converge to a value that is also not equal to unity. However, this expectation is for a fixed s and we need a corresponding result that holds uniformly for $\Re(s) > \sigma$.

We make some further observations about the expressions in (27) and (28). We assume $h > 0$ is fixed and chosen with $h \leq \hat{h}$, so that $|\kappa_0| < 1$.

- (1) $\int_0^\infty e^{-ts}k(t) dt \rightarrow 0$ as $|s| \rightarrow \infty$ along any path with $\Re(s) > \sigma > 0$. This follows by the Riemann–Lebesgue Lemma (for $\Im(s) \rightarrow \pm\infty$) and by the weak-order property of Laplace transforms (see [4, p. 190]) for $\Re(s) \rightarrow +\infty$.
- (2) The value of \mathcal{L}_h is unchanged by addition of integer multiples of $2\pi i$ to sh and so as $\zeta \rightarrow 0$, (or, equivalently, as $\Re(s) \rightarrow \infty$) $\mathcal{L}_h \rightarrow 0$ uniformly in $\Im(s)$.
- (3) \mathcal{L}_h is a regular function of s since the sequence $\{\kappa_n\} \in \ell^1$. This implies that the roots of the equation $\mathcal{L}_h(s) = 1$ are isolated.

For each fixed $h > 0$ it follows from these observations that there exists a fixed value $\mu_h \geq \sigma$ for which the $\mathcal{L}_h(s) \neq 1$ for any $\Re(s) > \mu_h$. We define $v_h = \sup_{\tilde{h} \leq h} (\mu_{\tilde{h}})$. Thus, as $h \rightarrow 0$, v_h is a non-increasing sequence of real numbers, bounded below by σ , with limit which we shall call v . We prove that $v = \sigma$.

Let $h > 0$ and let $\alpha > 0$ be some suitable constant and define $S_h = \{z : |\mathcal{L}_{\tilde{h}}(z) - 1| < \alpha h \text{ for some } \tilde{h} : 0 < \tilde{h} \leq h\}$. It follows that, for $0 < h_2 < h_1$, $S_{h_2} \subset S_{h_1}$.

Now we let $\{h_n\}$ be a decreasing sequence of step lengths tending to zero and we define $S = \bigcap_{i=1}^\infty S_{h_i}$. Using the piecewise continuity of k and the convergence of the quadrature rules, it follows that $\int_0^\infty e^{-tz}k(t) dt = 1$ (as the limit, as $h \rightarrow 0$ of $\mathcal{L}_h(z)$) for any value of $z \in S$.

By hypothesis, there are no values of z for which $\int_0^\infty e^{-tz}k(t) dt = 1$ for $\Re(z) > \sigma$ and so it follows that $v = \sigma$ as required.

We summarise this in the following:

Proposition 5.1. *Suppose a consistent and zero-stable linear multistep method with constant step size $h > 0$ is applied to an equation of form (3) with kernel satisfying hypothesis (H1). Then the exact solution is of exponential type σ and there is a constant $v_h \approx \sigma$ such that the approximate solution is of exponential type $\tilde{\sigma}$ for every $\tilde{\sigma} \geq v_h$. Further, as $h \rightarrow 0$, $v_h \rightarrow \sigma$.*

6. Further analysis for positive definite kernels

As we remarked previously, Bochner's characterisation of positive definiteness provides a tool for analysis of the integral equation. We were able to conclude that if k, g satisfy hypotheses (H2a,2b,2c) then the solution y of (1) exists for every $\lambda : \Re(\lambda) < 0$ and satisfies $y(t) = \psi(t)e^{\sigma t}$ where $\psi \in L^1[0, \infty)$.

For discrete equations, the corresponding characterisation of positive definiteness is given by the Toeplitz Carathéodory Theorem. A detailed discussion of the case where $\sigma = 0$ is given in the paper by Lubich (see [19]). Here we shall adapt the results of that paper in a very simple way to deal with kernels that are positive definite of exponential type. In the previous section, under fewer hypotheses, we were able to show that the true exponential type of the solution was preserved by a suitable numerical method, in the limit as $h \rightarrow 0$. Here we are able to give some results that show that the exponential type of a solution may be preserved in the numerical scheme for all $h > 0$.

In (1) we assume (H2a,2b,2c) and additionally

- (H3) We use a convolution quadrature derived from a strongly stable linear multistep method with stability disc \mathcal{S} and weights $\{\omega_n\}$.

This leads to an equation of the form

$$y_n = f_n + h \sum_{j=0}^n \omega_{n-j} k_{n-j} y_j, \quad (29)$$

which can be written as

$$\mathfrak{Y}_n = \mathfrak{F}_n + \lambda h \sum_{j=0}^n \omega_{n-j} \mathfrak{K}_{n-j} \mathfrak{Y}_j, \quad (30)$$

where $y_n = \mathfrak{Y}_n e^{\sigma h n}$, $f_n = \mathfrak{F}_n e^{\sigma h n}$, $k_n = \mathfrak{K}_n e^{\sigma h n}$. By [19] Theorem 5.1, it follows that $\mathfrak{Y}_n \rightarrow 0$ whenever $\mathfrak{F}_n \rightarrow 0$ for every value of $\lambda h \in \mathcal{S}$. In other words the exponential type of the solution y_n is preserved for all $\lambda h \in \mathcal{S}$. For an A-stable method, the exponential type of the exact solution will be preserved for all $\Re(\lambda h) < 0$. Note that, for $\Re(\lambda) < 0$ this result will be true for all step lengths $h > 0$.

We summarise:

Theorem 6.1. For Eq. (1) under hypotheses (H2a,2b,2c) and H3 assume $h\lambda \in \mathcal{S}$. The solution $\{y_n\}$ satisfies

$$y_n e^{-\sigma h n} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (31)$$

7. Numerical results

In this final section we shall investigate the numerical solution of Eq. (11) over a long time interval. As we shall see, the numerical results reflect the analytical properties of the solutions that we have derived in the previous sections.

Specifically we shall examine the case of the equation

$$y(t) = 1 + t + \int_0^t \frac{s^{\mu-1}}{t^\mu} y(s) ds, \quad (32)$$

which has (infinitely many) exact solutions of the form $y(t) = -ct^{1-\mu} + t((\mu+1)/\mu) + (\mu/(\mu-1))$ as c varies. The analysis of Section 5 applies in this case. We shall consider the propagation of the numerical approximations to trajectories of the solution (respectively, for $c = 10, 20$) to (32) using the trapezium rule to derive the approximate solution. To specify a particular solution, we follow the exact solution over an initial interval $[0, 1]$ and then use the trapezium rule to follow the subsequent trajectory for $t > 1$. We chose a step length $h = 0.1$. The two trajectories will correspond to different exact solutions that will separate at a rate proportional to $t^{1-\mu}$ as $t \rightarrow \infty$. The results of this

Table 1
Difference between approximate trajectories as $t = nh$ varies

t	Separations for $c = 10, 20$		
	for $\mu = \frac{1}{2}$	for $\mu = \frac{3}{4}$	for $\mu = \frac{4}{5}$
1	10.0000	10.0000	10.0000
2	14.1421	11.8921	11.4870
3	17.3205	13.1607	12.4573
4	20.0000	14.1421	13.1951
5	22.3606	14.9535	13.7973
6	24.4948	15.6508	14.3097
7	26.4575	16.2658	14.7577
8	28.2843	16.8179	15.1572
9	30.0000	17.3205	15.5185
10	31.6228	17.7828	15.8489
11	33.1662	18.2116	16.1539
12	34.6410	18.6121	16.4375
13	36.0555	18.9883	16.7028
14	37.4166	19.3434	16.9522
15	38.7298	19.6799	17.1877
16	40.0000	20.0000	17.4110
20	44.7214	21.1474	18.2056
50	70.7107	26.5915	21.8672
100	99.9999	31.6228	25.1189

paper predict that the numerical scheme will provide approximate trajectories that will separate at a rate *approximately proportional* to $t^{1-\mu}$ as $t \rightarrow \infty$.

We give, in Table 1, very clear numerical evidence that this does indeed happen. These results are reproduced in other examples we have tried. We also obtained identical results by comparing the trajectories for $c = 30, 40$.

8. Conclusions

It is well known [19] that the use of an A -stable linear multistep method for a linear convolution integral equation preserves the stability and boundedness of the exact solution. These results go further and guarantee that, under appropriate conditions on the kernel and forcing term, an A -stable linear multistep method will preserve the true exponential growth rate of the exact solution. Furthermore, under weaker conditions, the true exponential growth rate will be recovered in the limit as step length $h \rightarrow 0$.

However, one important aspect of the theory remains open: among asymptotically stable solutions of Eq. (3) there exist those which tend to zero at an exponential rate. One might suppose that the application of an A -stable linear multistep method to such problems will lead to similar conclusions; however, this is not covered by the existing theory and merits further investigation.

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References

- [1] C.T.H. Baker, A perspective on the numerical treatment of Volterra equations, *J. Comput. Appl. Math.* 125 (2000) 217–249.
- [2] C.T.H. Baker, G.F. Miller (Eds.), *Treatment of Integral Equations by Numerical Methods*, Proceedings of the Symposium Held at Durham University, Durham, July 19–29, 1982, Academic Press, London, 1982.
- [3] H. Brunner, P.J. van der Houwen, *The Numerical Solution of Volterra Equations*, North-Holland, Amsterdam, 1986.
- [4] R.V. Churchill, *Operational Mathematics*, third ed., McGraw-Hill, Tokyo, 1972.
- [5] T. Diogo, J.T. Edwards, N.J. Ford, S.M. Thomas, Numerical analysis of a singular integral equation, *Appl. Math. Comput.* 167 (2005) 372–382.
- [6] J.T. Edwards, N.J. Ford, J.A. Roberts, Bifurcations in numerical methods for Volterra integro-differential equations, *Internat. J. Bifurcation Chaos* 13 (2003) 3255–3271.
- [7] J.T. Edwards, J.A. Roberts, On the existence of bounded solutions to a difference analogue for a nonlinear integro-differential equation, *Internat. J. Appl. Sci. Comput.* 6 (1999) 55–60.
- [8] S.N. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 1996.
- [9] N.J. Ford, C.T.H. Baker, J.A. Roberts, Nonlinear Volterra integro-differential equations—stability and numerical stability of θ -methods, *J. Integral Equations Appl.* 10 (1998) 397–416.
- [10] N.J. Ford, J.T. Edwards, J.A. Roberts, L.E. Shaikhet, Stability of a discrete nonlinear integro-differential equation of convolution type, *Stability Control Theory Appl.* 3 (2000) 24–37.
- [11] N.J. Ford, V. Wulf, The use of boundary locus plots in the identification of bifurcation points in numerical approximation of delay differential equations, *JCAM* 111 (1999) 153–162.
- [12] K. Frischmuth, N.J. Ford, J.T. Edwards, Volterra integral equations with non-Lipschitz nonlinearity, *Rostock. Math. Kolloq.* 51 (1997) 65–82.
- [13] G. Gripenberg, S.-O. Londen, O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
- [14] W. Han, Existence, uniqueness and smoothness results for second-kind Volterra equations with weakly singular kernels, *J. Integral Equations Appl.* 9 (1994) 365–384.
- [15] P. Henrici, Automatic computations with power series, *J. Assoc. Comput. Mach.* 3 (1956) 10–15.
- [16] W. Lamb, A spectral approach to an integral equation, *Glasgow Math. J.* 26 (1985) 83–89.
- [17] P. Lima, T. Diogo, An extrapolation method for a Volterra integral equation with weakly singular kernel, *Appl. Numer. Math.* 24 (1997) 131–149.
- [18] P. Lima, T. Diogo, Numerical solution of a non-uniquely solvable Volterra integral equation using extrapolation methods, *JCAM* 140 (2002) 537–557.
- [19] Ch. Lubich, On the stability of linear multistep methods for Volterra convolution equations, *IMA J. Numer. Anal.* 3 (1983) 439–465.
- [20] R.E.A.C. Paley, N. Wiener, *Fourier Transforms in the Complex Domain*, Providence, 1934.
- [21] T. Tang, S. McKee, T. Diogo, Product integration methods for an integral equation with logarithmic singular kernel, *Appl. Numer. Math.* 9 (1992) 259–266.
- [22] P.H.M. Wolkenfelt, The construction of reducible quadrature rules for Volterra integral and integro-differential equations, *IMA J. Numer. Anal.* 2 (1982) 131–152.